Some Properties of Atomic Lattices

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Abstract. In this short paper we prove some elementary properties of atomic lattices which are useful for reasoning in this area. Indeed, the proves are not very difficult but it may serve as a kind of formulary for atomic lattices. The document is intended to be updated from time to time.

1 Full Atomic Lattices

First we define a complete Boolean algebra as a structure $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ (see [1, 2] for details) where $(M, \sqsubseteq)$ is an ordered set with least and greatest elements $\bot$ and $\top$, resp., $\bigsqcup$ and $\bigsqcap$ are supremum and infimum with respect to $\sqsubseteq$, supremum distributes over arbitrary infima and vice versa, and $\bar{\ }$ is the complement satisfying the de Morgan's laws $\overline{\bigsqcup M'} = \bigsqcap\{\overline{m'} \mid m' \in M'\}$ and $\overline{\bigsqcap M'} = \bigsqcup\{\overline{m'} \mid m' \in M'\}$ for all $M' \subseteq M$. $\sqcup$ and $\sqcap$ serve as abbreviations for binary supremum and infimum, resp. We define the symbols $\sqsupseteq$, $\sqsubset$, $\not\sqsubseteq$ and $\not\sqsupseteq$ by $m \sqsupseteq n \;\Leftrightarrow_{df}\; n \sqsubseteq m$, $m \sqsubset n \;\Leftrightarrow_{df}\; m \sqsubseteq n \wedge m \neq n$, $m \not\sqsubseteq n \;\Leftrightarrow_{df}\; \neg m \sqsubseteq n$ and $m \not\sqsupseteq n \;\Leftrightarrow_{df}\; \neg m \sqsupseteq n$. An element $m \in M$ is called *non-bottom* if $m \neq \bot$. In this setting we make the following definition:

Definition 1.1. *Let $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ be a complete Boolean algebra. A non-bottom element $m^a \in M$ is called* atomic *if for all non-bottom $m \in M$ the implication $m \sqsubseteq m^a \Rightarrow m = m^a$ holds. The set of all atomic elements of $\mathcal{M}$ is denoted by* $\mathsf{atom}(\mathcal{M})$. *$\mathcal{M}$ is a* full atomic lattice *if $m = \bigsqcup\{m^a \in \mathsf{atom}(\mathcal{M}) \mid m^a \sqsubseteq m\}$ holds for all $m \in M$.*

As a convention, we will denote atomic elements always with a superscript a.

In a full atomic lattice the following properties hold:

Lemma 1.2. *Let $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ be a full atomic lattice, and consider arbitrary atoms $m^a, n^a \in \mathsf{atom}(\mathcal{M})$, arbitrary $m, n \in M$ and an arbitrary $M' \subseteq M$. Then the following properties hold:*

1. $m^a \sqcap m \neq \bot \Leftrightarrow m^a \sqcap m = m^a$
2. $m^a \sqcap n^a = \bot \Leftrightarrow m^a \neq n^a$
3. $m^a \sqsubseteq m \Leftrightarrow m^a \sqcap m \neq \bot$
4. $m^a \sqcap m = \bot \vee m^a \sqcap m = m^a$
5. $m^a \sqsubseteq \bigsqcup M' \Leftrightarrow \exists m' \in M' : m^a \sqsubseteq m'$
6. $\overline{M'} = \bigsqcup\{\hat{m} \mid \hat{m} \notin M'\}$

7. $m^a \sqsubseteq m \Leftrightarrow m^a \not\sqsubseteq \overline{m}$
8. $m \sqsubseteq n \wedge m^a \sqsubseteq m \Rightarrow m^a \sqsubseteq n \vee m^a \sqsubseteq m \sqcap \overline{n}$
9. $m \sqcap n = \bot \wedge n^a \sqsubseteq n \Rightarrow m \sqcup n^a \sqsupset m$
10. $m \not\sqsubseteq n \Rightarrow \exists o^a \in \mathsf{atom}(\mathcal{M}) : o^a \sqsubseteq m \wedge o^a \not\sqsubseteq n$

Proof.
1: " $\Rightarrow$ ": We have $m^a \sqcap m \sqsubseteq m^a$ by definition of $\sqcap$, and because $m^a \sqcap m$ is non-bottom by assumption we have $m^a \sqcap m = m^a$ due to atomicity of m^a.

" $\Leftarrow$ ": m^a is atomic and hence non-bottom, so the claim follows immediately.

2: " $\Rightarrow$ ": Assume that $m^a \sqcap n^a = \bot$ and $m^a = n^a$ hold. Then we have $m^a = m^a \sqcap m^a = \bot$, contradicting the atomicity of m^a.

" $\Leftarrow$ ": Assume that $m^a \neq n^a$ and $m^a \sqcap n^a = m' \neq \bot$ hold. Then we have w.l.o.g. $m' \neq m^a$ and $m' \sqsubseteq m^a$ which contradicts the atomicity of m^a due to $m' \sqsubseteq \bot$.

3: By lattice theory, we have $m^a \sqsubseteq m \Leftrightarrow m^a \sqcap m = m^a$, so the claim is an easy consequence of Part 1.

4: Assume $m^a \sqcap m = m'$ with $\bot \neq m' \neq m^a$. Then we have $\bot \sqsubset m' \sqsubset m^a$ which contradicts the atomicity of m^a.

5: " $\Rightarrow$ ": By lattice theory, the left side is equivalent to $m^a \sqcap \bigsqcup M' = m^a$, and by distributivity equivalent to $\bigsqcup\{m^a \sqcap m' \mid m' \in M'\}$. Part 4 shows that every infimum $m^a \sqcap m'$ equals $\bot$ or m^a, however, due to $m^a \sqsubseteq \bot$ there has to be an $m' \in M'$ with $m^a \sqcap m' = m^a$, implying the claim.

" $\Leftarrow$ ": This direction is obvious by elementary lattice theory.

6: First we calculate as follows:

$$
\begin{array}{ll}
\bigsqcup\{m^a \mid m^a \in M'\} \sqcup \bigsqcup\{m^a \mid m^a \notin M'\} = & \{\text{ lattice theory }\} \\
\bigsqcup(\{m^a \mid m^a \in M'\} \cup \{m^a \mid m^a \notin M'\}) = & \{\text{ set theory }\} \\
\bigsqcup M = & \{\text{ lattice theory }\} \\
\top &
\end{array}
$$

On the other hand, we have the following:

$$
\begin{array}{ll}
\bigsqcup\{m^a \mid m^a \in M'\} \sqcap \bigsqcup\{m^a \mid m^a \notin M'\} = & \{\text{ distributivity }\} \\
\bigsqcup\{m^a_1 \sqcap m^a_2 \mid m^a_1 \in M', m^a_2 \notin M'\} = & \{\ m^a_1 \neq m^a_2, \text{Part 2 }\} \\
\bigsqcup\{\bot\} = & \{\text{ obvious }\} \\
\bot &
\end{array}
$$

This shows that $\bigsqcup\{\hat{m} \mid \hat{m} \notin M'\}$ fulfills the requirements of $\overline{M'}$.

7: This is an immediate consequence of Parts 5 and 6.

8: From $n \sqsubseteq m$ we conclude $m = m \sqcup n$ and hence $m = (m \sqcup n) \sqcap (\overline{n} \sqcup n) = (m \sqcap \overline{n}) \sqcup n$, so the claim follows from Part 5.

9: Clearly we have $m \sqcup n^a \sqsupseteq m$ so we have only to rule out equality, so let us assume that $m \sqcup n^a = m$ holds. Then we have $m \sqcap n = (m \sqcup n^a) \sqcap n = (m \sqcap n) \sqcup (n \sqcap n^a) = n^a$ by assumption, distributivity, lattice theory and Part 4. However, $m \sqcap n = n^a$ contradicts the premise $m \sqcap n = \bot$ by atomicity of n^a.

10: Assume that for all atomic o^a with $o^a \sqsubseteq m$ also the inequality $o^a \sqsubseteq n$ holds. Then we have $m \sqsubseteq n$ by atomicity of $\mathcal{M}$ and lattice theory which contradicts the initial assumption. ∎

References

1. Birkhoff, G.: Lattice Theory. Amer. Math. Soc., 3rd edn. (1967)

2. Jipsen, P., Rose, H.: Varieties of Lattices. Springer, first edn. (1992)